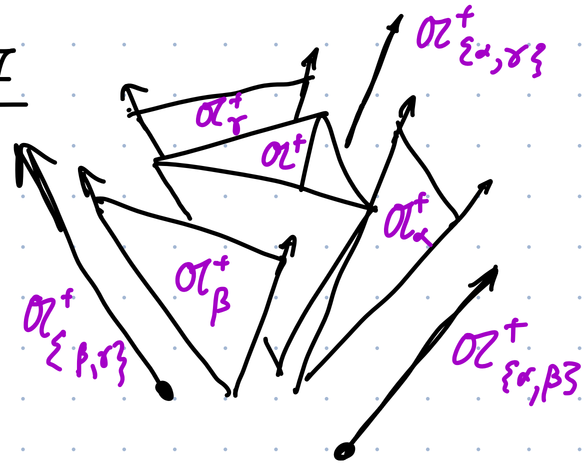
Lecture 37



$A_3 / SL_4\mathbb{R}$: $\mathfrak{a} \cong \mathbb{R}^3$ $\Delta = \{\alpha, \beta, \gamma\}$

$\mathfrak{a}^+ =$ cone over open triangle.

W acts on $\mathfrak{a}$ with $\overline{\mathfrak{a}^+}$ as a fund domain: Every orbit hits $\overline{\mathfrak{a}^+}$.
No orbit hits $\mathfrak{a}^+ = \text{int}\, \overline{\mathfrak{a}^+}$ twice.

If fact $W \curvearrowright \mathfrak{a}$ and each orbit hits $\overline{\mathfrak{a}}^+$ in single point.
It just may have stabilizer.

Thus every point in ${}^{\text{ve}}\mathfrak{a}$ has a "type", a subset $S \subset \Delta$ s.t. $W \cdot v \cap \mathfrak{a}^+_S \neq \phi$.

For an open dense set, the type is $S = \phi$.

<u>Thm</u>. $\text{Stab}_G([\gamma]) = P_S$ if $\gamma(t) = \exp(tv) \cdot x_0$ for vec of type S.

Note. $K \curvearrowright G/K$ fixing x_0 and acting transitively on $T_{x_0} G/K \cong \mathfrak{p}$.
So the theorem above is effectively the general case.

<u>Distance formulas</u>. $x, y \in G/K$.

$d(x,y)$? WLOG $x = x_0$ $y = g x_0$. $g = k_1 \exp(a) k_2$
where $a \in \mathfrak{a}$.

Then since $(\mathfrak{a}, \| \, \|_B) \xrightarrow{f} G/K$ is an isometry (tot geod sub),
$a \longmapsto k_1 \exp(a) k_1^{-1} x_0$

$d(x_0, y) = d(f(0), f(a)) = \|a\|_B$ $\quad (:= B(a,a)^{1/2})$

Recall a is not unique but $\|a\|_B$ is! Alternatively.

<u>Def</u>. Cartan projection $\mu : G \to \overline{\mathfrak{a}}^+$. Def by — Facts: Surjective, Continuous, Smooth a.e.

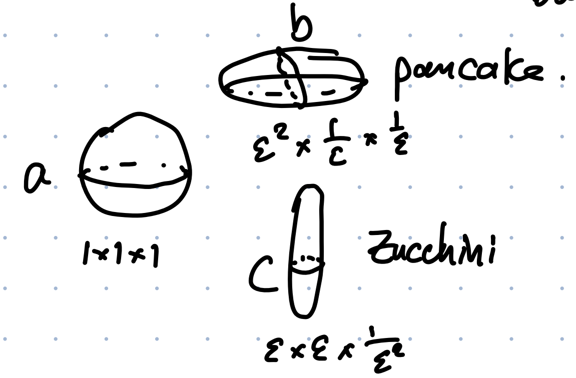
$g = k_1 \exp(\mu(g)) k_2$. So for $SL_n\mathbb{R}$, $\mu(g) = \begin{pmatrix} \log \sigma_1 & & \\ & \ddots & \\ & & \log \sigma_n \end{pmatrix}$

with σ_i the singular values in decreasing order.

<u>Refined distance</u>. $\vec{d}(gx_0, hx_0) = \mu(g^{-1}h) \in \overline{\mathfrak{a}}^+$ $\quad d(x,y) = \|\vec{d}(x,y)\|_B$

aka Weyl chamber valued

i.e. not just "how different are these ellipsoids?" but also "is the difference in major axis? minor? ..."



$d(a,b) = d(a,c)$ but $\vec{d}(a,b) \neq \vec{d}(a,c)$

<u>Sketch of thm</u>. Stab is a closed subgrp. Let's find its Lie alg.

Let $g \in G$ have form $g = \exp(x)$.

$$\vec{d}(\gamma(t), g\cdot\gamma(t)) = \vec{d}(\exp(tv), \exp(x)\exp(-tv))$$
$$= \mu(\exp(\mathrm{Ad}_{\exp(-tv)} x))$$

Write x as $x = \underbrace{x_0}_{\mathfrak{a},\, \mathfrak{m}} + \sum_\alpha x_\alpha$. $\quad \mathrm{Ad}_{\exp(-tv)} x_0 = x_0$

$$\mathrm{Ad}_{\exp(-tv)} x_\alpha = e^{-t\alpha(v)} x_\alpha.$$

Now $v \in \overline{\mathfrak{a}}^+$. So $\alpha(v) \geq 0$ for all the positive roots α.

So any $x_\alpha \in \mathfrak{n}$ gives exp decay.

x_α for negative root α will give exp growth unless $\alpha(v) = 0$

That is, x can have neg root cpnts only in span(S) where $S = \mathrm{type}(v)$.

(The fact that exp growth really gives dist $\to \infty$ needs some work)

$\square$

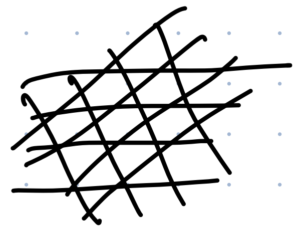
So $\forall\ P$, $G/P \hookrightarrow \partial_{vis} G/K$.

Dimension count: $G = SL_3\mathbb{R}$ (8) $\quad K = SO(3)$ (3)

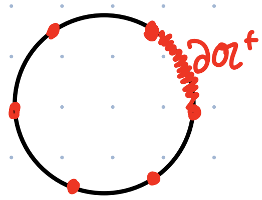
$G/K \cong \mathbb{R}^5 \quad \partial_{vis}(G/K) \cong S^4$

But G/P is either $\underbrace{Flag(\mathbb{R}^3)}_{3}$ or $\underbrace{\mathbb{RP}^2}_{2}$.

How are these put together? $\quad \mathfrak{a} =$

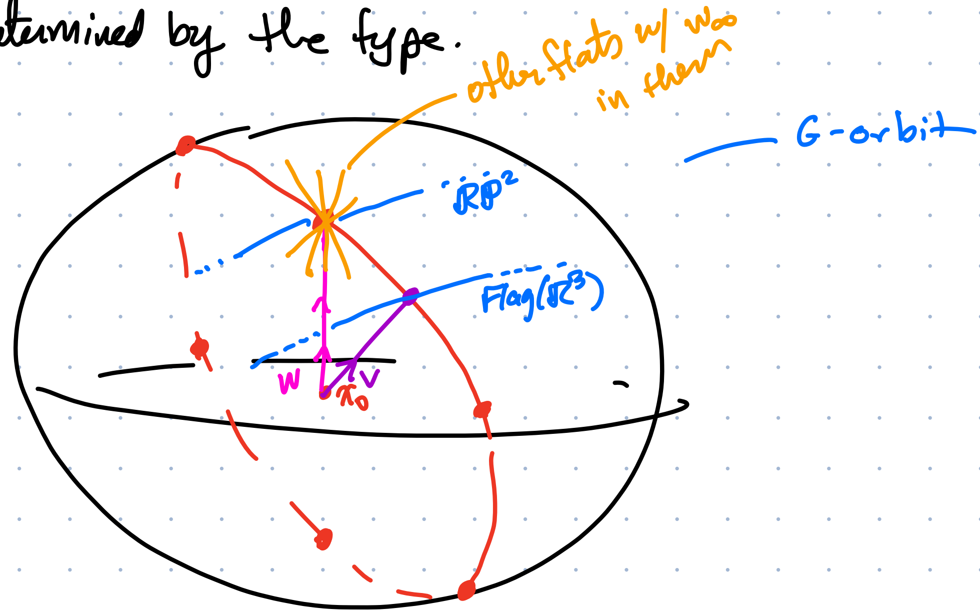


$\partial_\infty \mathfrak{a} =$



spherical coxeter complex.
($\|v\|=1$ in $\mathfrak{a}$ w/ W action)

Every geodesic lies in a flat. Its orbit is a G/P where P is determined by the type.



This gives $\partial_{vis} G/K$ the structure of a spherical building:
A space + a collection of subspaces called apartments.
Each apartment should have a simplicial complex structure iso to spherical coxeter cplx (fixed).
Intersections of apartments should be subcomplexes.